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An experiment on third-order resonant wave interactions

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An experiment has been carried out to verify the existence of the resonant interaction between trains of gravity waves, predicted by Phillips (1960). As suggested by Longuet-Higgins (1962), two trains of waves in mutually perpendicular directions were generated in a rectangular wave tank. The ratio σ_1/σ_2 of the wave frequencies was varied ($1\cdot4 < \sigma_1/\sigma_2 < 2\cdot1$). When $\sigma_1/\sigma_2 \doteqdot 1\cdot7357$ it was expected that a resonant interaction would take place, generating a wave of frequency $(2\sigma_1 - \sigma_2)$. The amplitude of the third wave was expected to increase almost linearly in the direction of wave propagation. The shape of the response curve as a function of σ_1/σ_2 was also predicted.

In the present experiments rather large wave amplitudes had to be used, and the theoretical shape of the response curve was distorted by non-linear detuning. Nevertheless the peak amplitude of the resonant wave was found to increase with distance in very nearly the manner predicted.

These experiments were carried out in 1961 but publication was deferred pending a similar but more accurate investigation by McGoldrick, Phillips, Huang & Hodgson (1966). Much of the theoretical discussion given in the present paper is relevant to their work.

1. Introduction

It was first shown by Phillips (1960) that three trains of gravity waves in deep water, with horizontal wave-numbers $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$, say, may under certain conditions interact so as to transfer energy to a fourth wave-number, say $\mathbf{k}_4$. Two necessary conditions for such interaction are that

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 \pm \mathbf{k}_4 = 0, \tag{1}$$

and

$$\sigma_1 \pm \sigma_2 \pm \sigma_3 \pm \sigma_4 = 0, \tag{2}$$

where σ_i denotes the frequency† corresponding to $\mathbf{k}_i$. Hasselmann (1962) has shown that two of the signs in (1) must be positive and two negative; and similarly in (2).

Any two of the wave numbers, say $\mathbf{k}_1$ and $\mathbf{k}_3$, may also be equal (Phillips 1960) in which case one has

$$\left.\begin{aligned} 2\mathbf{k}_1 - \mathbf{k}_2 &= \mathbf{k}_4, \\ 2\sigma_1 - \sigma_2 &= \sigma_4. \end{aligned}\right\} \tag{3}$$

The locus of $\mathbf{k}_2$ is then a certain figure-of-eight curve, with $\mathbf{k}_1$ at the centre (see Phillips 1960, figure 1).

† Radian frequencies and angular wave-numbers are used throughout this paper.

If the three wave amplitudes a_1, a_2, a_3 are small, and if a_4 is initially zero, then Phillips showed that a_4 is proportional to the product $a_1 a_2 a_3$ at first, and it grows proportionally to the time t. Benney (1962) considered a more general situation when all four wave amplitudes $a_1, \ldots, a_4$ were of comparable magnitude. The existence of periodic solutions in this case was shown by Bretherton (1964).

In a series of papers (1960, 1962, 1963*a*, *b*), Hasselmann has studied the resonant transfer of energy for a continuous wave spectrum, and finds (1963*b*) that for ocean waves the modification of the wave spectrum by this mechanism should be appreciable.

On the other hand, the whole analysis has been called in question by Pierson (1961), who has suggested that the apparent transfer of energy is not real, but is a consequence of a deficiency in the method of approximation.

In order to test the theory under controlled conditions one of the present authors (Longuet-Higgins 1962) suggested a simple experiment that could be carried out in a fairly small wave tank. Let two wave-makers be placed on adjacent sides of a rectangular tank, with wave absorbers on the two sides opposite (as in figure 1.). Let one of the wave-makers generate waves with frequency σ_1 and the other with frequency σ_2. The wave-numbers $\mathbf{k}_1$ and $\mathbf{k}_2$ are mutually perpendicular. Then when the ratio approaches a certain value such that the conditions (3) are satisfied it should be possible to detect a wave of frequency $(2\sigma_1 - \sigma_2)$ and wave-number $(2\mathbf{k}_1 - \mathbf{k}_2)$ due to the resonant interaction. The rate of growth was calculated theoretically in the paper just mentioned (Longuet-Higgins 1962).

The present paper is an account of an attempt to carry out this experiment. The observations were made during 1961 and 1962 at the Admiralty Experimental Works, Haslar, and were described by the present authors in a preliminary report dated 1962. The results, as will be seen, did indeed show the existence of a resonant interaction, but at the large wave slopes used, the shape of the response curve was distorted by non-linear effects (see below). Since the wave amplitude was limited by the sensitivity of the apparatus, publication was deferred, while, under the guidance of Dr Phillips, similar but more precise investigations were begun at Johns Hopkins University. These are described fully in an adjoining paper (McGoldrick *et al.* 1966), and appear to have also verified the theoretical form of the response curve. Our own experiments remained unpublished till now. Nevertheless, McGoldrick, *et al.* made some use of our results and calculations, and have referred to them. It seems convenient to present them here. Moreover these earlier experiments, though less conclusive in themselves, may have some additional points of interest to recommend them.

It should perhaps be mentioned that a different type of non-linear interaction between gravity waves has recently been observed and analysed by Benjamin & Feir (1966).

2. Theoretical expectations

In this section we shall summarize the main theoretical results that are relevant to the subsequent experiments.

The resonance condition. The two primary† wave trains will be supposed to be mutually perpendicular, as in figure 1. If, to begin with, we assume that the waves are unaffected by finite depth or surface tension then the resonance condition can be derived very simply as follows. (Afterwards we take these affects into account, as well as the effect of finite wave amplitude.)

Since the primary waves $\mathbf{k}_1$, $\mathbf{k}_2$ are both free waves (to first order) we have

$$\left.\begin{aligned}\sigma_1^2 &= gk_1,\\ \sigma_2^2 &= gk_2,\end{aligned}\right\} \tag{2.1}$$

where $k_i = |\mathbf{k}_i|$ and g denotes the acceleration of gravity. In order that the interaction wave $\mathbf{k}_4$ shall also be a free wave we must have

$$(2\sigma_1 - \sigma_2)^2 = g|2\mathbf{k}_1 - \mathbf{k}_2|. \tag{2.2}$$

In the special case when the two primary waves are mutually perpendicular, this condition becomes

$$(2\sigma_1 - \sigma_2)^2 = g(4k_1^2 + k_2^2)^{\frac{1}{2}} = (4\sigma_1^4 + \sigma_2^4)^{\frac{1}{2}}, \tag{2.3}$$

from (2.1). Let us write for convenience

$$\sigma_1/\sigma_2 = r. \tag{2.4}$$

Then on rejecting the trivial case $\sigma_2 = 0$, we obtain from (2.3)

$$(2r-1)^2 = (4r^4+1)^{\frac{1}{2}}. \tag{2.5}$$

Inspection shows that this equation has just two roots: $r = 0$ and $r = 1{\cdot}7357$ $(\doteqdot \sqrt{3})$. Thus there is just one non-trivial solution, namely

$$\sigma_1/\sigma_2 = 1{\cdot}7357 = r_0, \tag{2.6}$$

say. The ratio of the wave-numbers is then

$$k_1/k_2 = 3{\cdot}0123, \tag{2.7}$$

so that the wave-number $\mathbf{k}_4$ makes an angle with k_1 given by

$$\tan^{-1}(k_1/2k_2) = 9^\circ\, 24' \tag{2.8}$$

(see figure 1).

We have so far neglected the effects of surface tension, finite depth and finite wave amplitude (including the mutual interaction of the waves). It can be shown (see Appendix 1) that provided each of these effects is small, resonance occurs when

$$\sigma_1/\sigma_2 = r_0 + \Delta r = r_0', \tag{2.9}$$

say, where r_0 is given by (2.6),

$$\Delta r = -1{\cdot}957\,\Sigma\epsilon_1 - 0{\cdot}054\Sigma\epsilon_2 + 2{\cdot}011\Sigma\epsilon_4, \tag{2.10}$$

and ϵ_1, ϵ_2, ϵ_4 are the quantities summarized in table 1. The summation Σ refers to the sum of the corrections in each column. Thus Δr represents the total amount whereby the frequency ratio is detuned.

† It is convenient to refer to the waves with wave-numbers $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ as the *primary* waves, and the wave with wave-number $\mathbf{k}_4$ as the tertiary wave. This distinction can only be made when the amplitude a_4 is initially zero.

	ϵ_1	ϵ_2	ϵ_4
Finite depth	$-2e^{-2k_1h}$	$-2e^{-2k_2h}$	$-2e^{-2k_4h}$
Surface tension	$Tk_1^2/(\rho g)$	$Tk_2^2/(\rho g)$	$Tk_4^2/(\rho g)$
Finite amplitude	$1{\cdot}00a_1k_1^2$	$-0{\cdot}07a_1^2k_1^2$	$2{\cdot}92a_1^2k_1^2$
	$-0{\cdot}66a_2^2k_2^2$	$1{\cdot}00a_2^2k_2^2$	$-2{\cdot}15a_2^2k_2^2$
	$0{\cdot}69a_4^2k_4^2$	$-0{\cdot}06a_4^2k_4^2$	$1{\cdot}00a_4^2k_4^2$

TABLE 1

The corrections for finite amplitude have been calculated on the assumption that the motion is irrotational, which is known to be true only in the initial stages of the motion (Longuet-Higgins 1960). Generally, some second-order vorticity will be diffused into the interior, producing currents of order $a_1^2\sigma_1k_1$ or $a_2^2\sigma_2k_2$. These will affect the critical ratio r to an extent comparable with the effects mentioned. However, since the mass-transport currents are somewhat unpredictable, their effect is not included.

Amplitude of the tertiary wave. Under conditions of resonance, the expected wave amplitude was shown by Longuet-Higgins (1962) to be given by

$$a_4 = 0{\cdot}442(a_1k_1)^2(a_2k_2)\,x, \tag{2.11}$$

where x denotes the horizontal distance from wave-maker 1, in the direction of propagation. That is to say, the wave amplitude grows (at first) linearly with horizontal distance. If viscous dissipation is taken into account (see Appendix 2), equation (2.11) must be replaced by

$$a_4 = 0{\cdot}442\,(a_1k_1)^2(a_2k_2)(1-e^{-Bx})/B, \tag{2.12}$$

where

$$B = 4\nu\sigma_4^5/g^3. \tag{2.13}$$

This turns out to be a fairly small correction.

A natural procedure is to observe the amplitude of the resonant wave at different distances from the wave-maker and under conditions when the frequency ratio r is near to (but not necessarily at) resonance. The shape of the response curve as a function of r, is considered in Appendix 2. It is shown there that the expected amplitude a_4 is given approximately by

$$a_4 = a_{4\max}\,|\sin(x\delta k)/(x\delta k)|, \tag{2.14}$$

where $a_{4\max}$ is the value given by equation (2.12) and where

$$\delta k = -0{\cdot}249k_4(r-r_0'). \tag{2.15}$$

Thus the width W of the response curve, that is to say the interval of r separating the peak response from the nearest minimum value, is given by

$$x\delta k = \pi, \quad W = 12{\cdot}7/(k_4x). \tag{2.16}$$

Hence the width is inversely proportional to the distance from wave-maker 1.

The above formulae take no account of the detuning effects mentioned earlier, except in so far as r_0 is replaced by r_0'.

If the distance x were allowed to increase indefinitely, the finite value of the steepness a_4k_4 of the tertiary wave would ultimately shift the resonant frequency through a distance comparable with the central region of the response curve. The above formulae would then not apply. However, in the present experiments the detuning due to a_4k_4 was small compared with W, so that the necessary condition was satisfied.

There must also be some transfer of energy from $\mathbf{k}_1$ to both $\mathbf{k}_2$ and $\mathbf{k}_4$, leading to a slight change in the amplitudes a_2 and a_4. But these changes were also negligible.

3. Description of the apparatus

The experiments were carried out in a rectangular tank, of which a plan view is shown in figure 1. The overall horizontal dimensions were 10 ft. by 8 ft., (3·05 m by 2·44 m) and the maximum possible depth was 9·0 in. (23 cm). The two wave-makers, shown schematically in figure 1, were of the vertical plunger type, having a wedge-shaped vertical cross-section. Each was driven by a 1 h.p. electric motor. By using a battery power supply, the motor speeds could be kept constant to within 0·2 % over a period of 3 min.

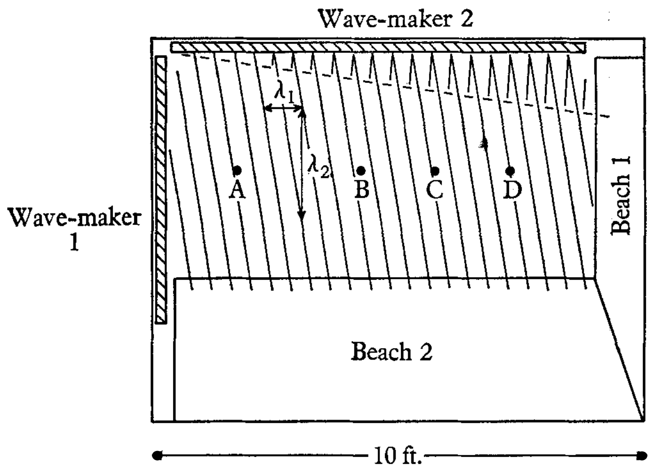

FIGURE 1. Plan of the wave tank, showing the theoretical crest-pattern of the tertiary waves, and the zone of reflexion. The surface displacement was recorded at A, B, C and D. λ_1 and λ_2 denote the wavelengths of the two primary waves.

On the sides of the tank opposite the wave-makers there were wave-absorbing beaches, so designed as to reflect not more than 1 % of the wave energy. Since wave-maker 1 was always operated at a higher speed than wave-maker 2, the corresponding beach could be narrower.

According to the simple theory given in § 2 the direction of the tertiary interaction wave at resonance should be at an angle of 9° 24′ with the direction of wave 1 (away from the corresponding wave-maker) and with a slight component towards wave-maker 2. Hence there should be a narrow zone of reflexion from wave-maker 2, as indicated in figure 1.

One expects the amplitude of the teritary wave to increase proportionally with distance in the direction of wave propagation. Continuous records of the surface elevation were therefore made at each of the points A, B, C and D in figure 1, at distances of 1·5, 4·0, 5·5 and 7·0 ft., (46, 124, 155 and 216 cm) respectively from wave-maker 1, measured parallel to the direction of k_4, and 2·5 ft (76 cm) from wave-maker 2. These positions are well outside the zone of reflexion.

The wave recorder used was of the capacitance-wire type, having two parallel vertical elements about 0·65 cm apart. Surface-tension effects, at first troublesome, were overcome by treating the wires with a detergent. The apparatus was calibrated by raising and lowering the wires through a known distance.

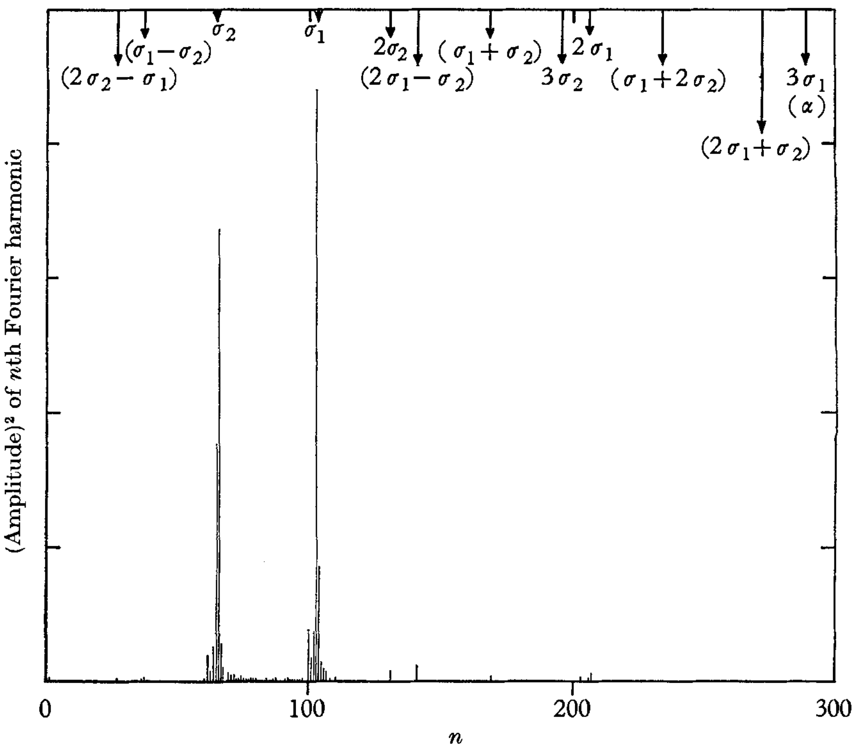

FIGURE 2. Complete harmonic analysis of a typical record, at position C.

After amplification, the electrical signal from the wave recorder was fed to one channel of a high-speed pen recorder. The remaining channels of the recorder were used for time markings. A microswitch on the shaft of each wave-maker made a contact once every revolution, giving an accurate measure of the relative frequencies. A third time-trace was operated by a one-minute time switch.

4. Procedure and method of analysis

In operation, the frequency σ_1 of wave-maker 1 was kept constant and the frequency σ_2 was altered in small steps. At each setting of σ_2, time was allowed for the wave conditions to become steady. In this way the whole range

$$1{\cdot}4 \leqslant \sigma_1/\sigma_2 \leqslant 2{\cdot}1$$

was examined, at each of the stations A, B, C and D.

The record of the surface elevation at any point contains, besides the two fundamental frequencies σ_1 and σ_2 and a small amount of random noise, also harmonics of the form $(n\sigma_1 + m\sigma_2)$, where n and m are positive or negative integers or zero. To examine the output in a typical case, when $\sigma_1/\sigma_2 = 1{\cdot}58$, the recorded output was digitized and a harmonic analysis performed by a standard routine.† The squares of the Fourier components (regardless of phase) are as shown in figure 2. From this it can be seen that the components σ_1 and σ_2 are by far the largest in the record, and that the next largest is the interaction frequency $(2\sigma_1 - \sigma_2)$, closely followed by the second harmonics $2\sigma_1$ and $2\sigma_2$. The remaining harmonics are all considerably smaller. The magnitude of the second harmonics is such as would be expected on the ordinary second-order theory, in which resonance plays no part. The magnitude of the resonance harmonic will be discussed below.

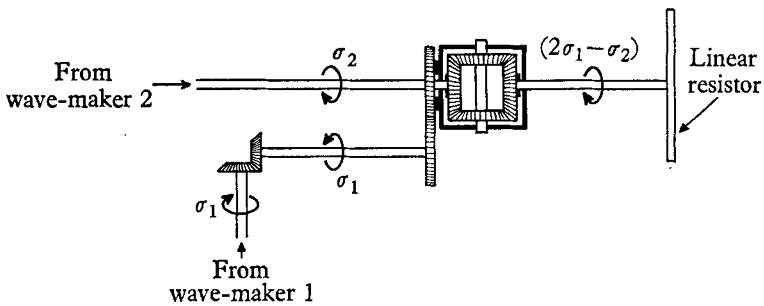

FIGURE 3. Diagram of the differential gear driving the harmonic analyser.

However, to examine each frequency spectrum in detail for the large number of records required, would have been unnecessarily laborious, since in general only the magnitude of the harmonic $(2\sigma_1 - \sigma_2)$ was required. Accordingly a very simple analogue harmonic analyser, for this particular frequency, was devised as follows.

Consider the differential gear shown in figure 3. The gear is normally driven by rotating the cage at an angular frequency σ_1, say. If the two other shafts are so constrained as to rotate at the same speed, then both will rotate with angular speed σ_1. But if one of the shafts is held stationary, the other will rotate at a speed $2\sigma_1$.

On the other hand, if the *cage* is held still and one of the other shafts is rotated at a speed σ_2, the remaining shaft will rotate with speed $-\sigma_2$. The motions are additive. Hence if the cage is driven with speed σ_1, and one of the other shafts with speed σ_2, the remaining shaft will rotate with the required speed $(2\sigma_1 - \sigma_2)$.

Making use of this principle, the driving shaft of wave-maker 1 was coupled directly to the cage of a differential gear; the driving shaft of wave-maker 2 was coupled to one arm of the differential. The other arm of the differential was then made to rotate a linear, wire-wound resistor, across which the electrical output $f(t)$ of the wave probe, was applied through moving electrical contacts. A

† BOMM.

stationary pair of contacts then picked up the voltage across the resistor. This was proportional to $f(t)\cos\theta$, where

$$\theta = (2\sigma_1 - \sigma_2)(t - t_0)$$

denotes the orientation of the resistor. The output was then smoothed by feeding through an R.C. filter with a time-constant of 30 sec. The smoothed output then gave a measure of the Fourier cosine component at frequency $(2\sigma_1 - \sigma_2)$. A second pair of contacts, at right-angles to the first, gave a measure of the Fourier sine component. If these components are denoted by C and S respectively, then the total amplitude of the harmonic was taken to be $(C^2 + S^2)^{\frac{1}{2}}$.

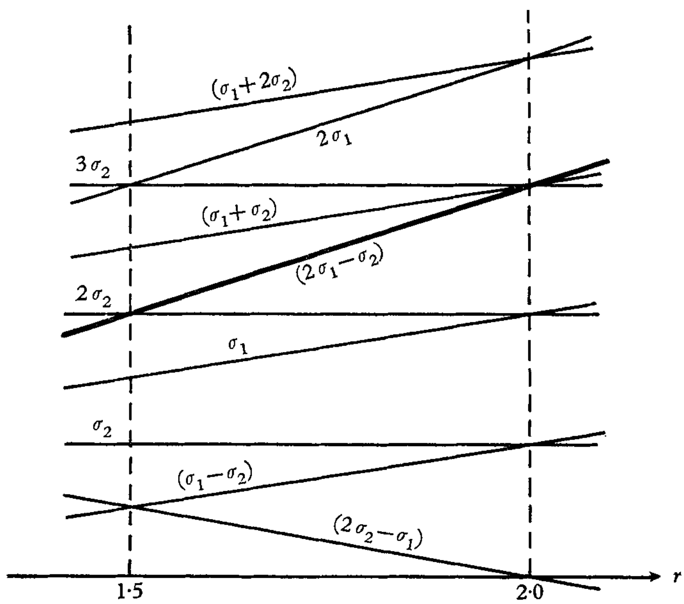

FIGURE 4. The relation between the harmonics $(n\sigma_1 + m\sigma_2)$ in the range $1{\cdot}5 \leqslant r \leqslant 2{\cdot}0$.

The amplitude determined in this way was compared with the amplitude determined by the complete harmonic analysis in the special case mentioned above. The two determinations of the amplitudes by the two methods were very nearly identical. Accordingly, in general the simpler analogue method was used.

The analysis using the differential gear had also the advantage that it referred the tertiary wave to the actual frequencies of the primary waves rather than to a sine wave of fixed frequency, and so was less sensitive to frequency drift.

Figure 4 shows the frequencies of the harmonics $(n\sigma_1 + m\sigma_2)$ as a function of the ratio $r = \sigma_1/\sigma_2$. It can be seen that at two points in the range, namely $r = 1{\cdot}5$ and $r = 2{\cdot}0$, the frequency of interest, namely $(2\sigma_1 - \sigma_2)$, coincides with another of the harmonics. At these points it was not possible to determine the amplitude a_4. However, when the ratio r was quite close to 1·5 or 2·0 it was still possible to measure a_4, since the effect of the unwanted harmonics was simply to produce slow beats in the output of the R.C. filter. The amplitude of the harmonic $(2\sigma_1 - \sigma_2)$ may then be shown to correspond to the mean value of the output.

5. Experimental parameters

The sensitivity of the apparatus was limited by the wave probe itself. In order that the amplitude a_4 of the tertiary wave should grow to an appreciable value within the limited space of the tank, it was necessary that the steepness of the primary waves, especially $(a_1 k_1)$, should be as large as possible.

The stroke of the wave-makers was kept constant at 0·5 in. (1·27 cm). The wave amplitude a_1 was constant at about 1·06 cm. The amplitude a_2 depended somewhat on the frequency, diminishing from 0·85 cm at $r = 1{\cdot}5$ to about 0·70 at $r = 2{\cdot}0$. Near the resonance peak, a_2 was 0·74 cm. Thus, for the wave amplitudes we have

$$a_1 = 1{\cdot}06\,\text{cm}, \quad a_2 \simeq 0{\cdot}74\,\text{cm}.$$

The (constant) parameters for wave 1 were given by

$$\left.\begin{aligned} \sigma_1 &= 2\pi/0{\cdot}38 = 16{\cdot}5\,\text{sec}^{-1}, \\ k_1 &= \sigma_1^2/g = 0{\cdot}28\,\text{cm}^{-1}, \\ a_1 k_1 &= 0{\cdot}281. \end{aligned}\right\}$$

The parameters for wave 2 were variable, but at the resonance frequency σ_2, when $\sigma_1/\sigma_2 \doteqdot 1{\cdot}92$, we have

$$\left.\begin{aligned} \sigma_2 &= \sigma_1/1{\cdot}92 = 8{\cdot}6\,\text{sec}^{-1}, \\ k_2 &= \sigma_2/g = 0{\cdot}076\,\text{cm}^{-1}, \\ a_2 k_2 &= 0{\cdot}056. \end{aligned}\right\}$$

Corresponding to these values we have

$$\left.\begin{aligned} \sigma_4 &= (2\sigma_1 - \sigma_2) = 24{\cdot}4\,\text{sec}^{-1}, \\ k_4 &= (4k_1^2 + k_2^2)^{\frac{1}{2}} = 0{\cdot}56\,\text{cm}^{-1}. \end{aligned}\right\}$$

The expected amplitude a_4 of the tertiary wave was of the order of a few mm.

It may be as well to emphasize how critically the amplitude a_4 may be expected to depend on the frequency σ_1, for a given ratio r. This is because a_4 is proportional to $(a_1 k_1)^2 (a_2 k_2)$; and since k_1 is nearly proportional to σ_1^2, and k_2 to σ_2^2, a_4 varies roughly as the sixth power of σ_1.

The amplitude a_4 will also be affected by any reflexion, however small, of wave energy from the beaches, For example, a wave $-\mathbf{k}_2$ reflected from beach 2, having only 10 % of the amplitude of $\mathbf{k}_2$, would produce a corresponding tertiary wave travelling in almost the same direction as that to be measured, and changing their combined amplitude by as much as 10 % depending on their relative phase at the point of observation. A second reflexion of $\mathbf{k}_2$ at wave-maker 2 could increase this effect to 20 %. Whether the effects will reinforce or cancel one another at a fixed point will depend on their relative phase, and hence on the wave-number $\mathbf{k}_2$, which in turn depends on the frequency.

6. Comparison of theory and observation

The observed amplitudes a_4 of the tertiary harmonic are plotted against the ratio r in figures 5(*a*) to 5(*d*). The measurements have been normalized by dividing by $(a_1 k)^2 (a_2 k_2)$.

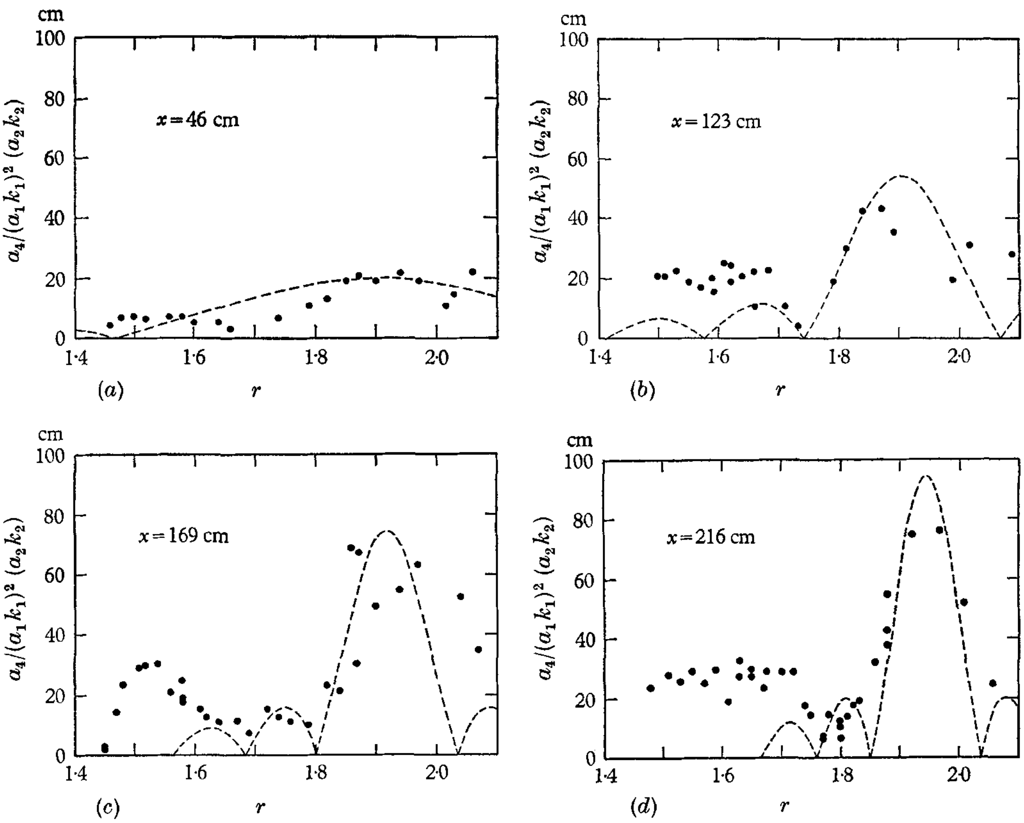

FIGURE 5(*a*) to (*d*). Observed values of $a_4/(a_1k_1)^2\,(a_2k_2)$, measured at positions A, B, C and D, respectively.

It can be seen that the main peak in each of the four figures occurs somewhat to the right of the uncorrected value $r_0 = 1{\cdot}736$. The maximum observed value of $a_4/(a_1k_1)^2(a_2k_2)$ in each of figures 5(*a*) to 5(*d*) is plotted against the distance x from wave-maker 1 in figure 6. Also in figure 6 is shown the theoretical peak amplitude given by the formula $0{\cdot}442x$. From this it can be seen that the peak amplitude increases with the distance x in about the same way as predicted.

The theoretical width W of the response curve, as given by (2.16) is shown below in table 2. k_4 has been kept constant at $0{\cdot}56\,\mathrm{cm}^{-1}$. Curves of the form (2.14) have been fitted roughly to the observations in figures 5(*a*) to 5(*d*). As can be seen, there is some resemblance to the observed distribution of points. However in figures 5(*c*) and 5(*d*) the observations show a noticeable rise in amplitude for values of r less than 1·6, in contrast to the theoretical prediction.

To estimate the expected positions of the peak amplitude we have inserted in table 1 the values of a_1, a_2, k_1, k_2 and h given above, leaving $(a_4 k_4)$ for the moment undetermined. The following corrections are obtained.

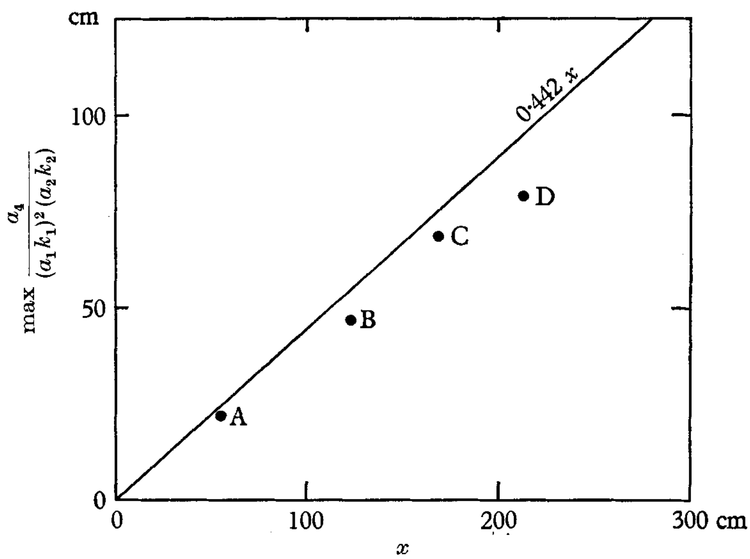

FIGURE 6. The largest observed values of $a_4/(a_1 k_1)^2 (a_2 k_2)$, at the four positions A, B, C, D. The solid curve represents the theoretical peak amplitude.

Position	x	W
A	46	0·48
B	124	0·18
C	155	0·15
D	216	0·11

TABLE 2

	ϵ_1	ϵ_2	ϵ_4
Finite depth	0·000	-0.054	0·000
Surface tension	0·006	0·001	0·030
Finite amplitude	0·076	0·002	0·221
	$+0.69\,(a_4 k_4)^2$	$-0.060\,(a_4 k_4)^2$	$+(a_4 k_4)^2$

TABLE 3

On substituting in equation (2.10) we obtain

$$\Delta r = 0{\cdot}35 + 0{\cdot}66(a_4 k_4)^2. \tag{6·1}$$

Hence, we should expect the critical ratio to be somewhat greater than

$$r_0 = 1{\cdot}736,$$

and to increase very slightly with $(a_4 k_4)$. Since $(a_4 k_4)$ is of the order of 0·2 or less, the latter correction is small.

Since the mass-transport currents have been neglected, equation (6.1) is valid only to within an order of magnitude. Nevertheless it may be noted that the displacement of the peak response to the right of r_0 is of the same order, as expected, though less by a factor of about 2.

7. Discussion and conclusions

In spite of the scatter of the observations in figure 5, it is apparent that the amplitude a_4 of the tertiary component, as exemplified by the peak amplitude in figure 6, does increase almost linearly with x in the manner predicted. From this alone it is legitimate to conclude that the resonant interaction has been observed.

Owing to the limited sensitivity of the wave probe, it was necessary to work with primary waves of rather large steepness, especially $a_1 k_1$. Hence it was not possible to verify convincingly the theoretical shape of the response curve as a function of r. Nevertheless, the shape of the response curve, as given by the width and mean position of the resonant peak, are certainly consistent with the approximate theory.

The problem of increasing the sensitivity of the wave probe has been solved by McGoldrick *et al.* (1966). They have thus been able to work with smaller steepness and to verify more precisely the form of the response curve.

We are indebted to the Director of the Admiralty Experimental Works, Mr R. M. Newton, for the use of the model tank in the A.E.W. Wave Laboratory. Generous assistance was given by Mr J. E. Connolly, Mr W. J. Wilkinson and other members of the A.E.W. Staff, to whom we express our thanks. Mr B. J. Hinde and Mrs P. Edwards of the National Institute of Oceanography kindly assisted us with the numerical analysis of the measurements. The preliminary report of these experiments was prepared while one of us (M. S. L.-H.) was visiting the Institute of Geophysics and Planetary Physics, University of California, in 1961 and 1962. He is indebted to the Director, Walter Munk, for his hospitality.

Appendix 1

Corrections to the resonant frequency ratio

We now calculate the corrections to the critical ratio (in other words the 'detuning') which results from three different effects: surface tension, finite depth of water and finite wave amplitude. In the last of these we include both the self-interactions and the mutual interactions of the primary waves. When each of the effects is small they may be calculated by the method of small perturbations, and are simply additive.

Suppose, in a general way, that the relation between frequency and wave-number for the two primary waves is given not by equations (2.1) but rather by

$$\left.\begin{aligned} \sigma_1^2 &= gk_1(1+\epsilon_1), \\ \sigma_2^2 &= gk_2(1+\epsilon_2), \end{aligned}\right\} \tag{A 1}$$

where ϵ_1 and ϵ_2 are small compared with unity. Suppose further that for the tertiary wave

$$(2\sigma_1 - \sigma_2)^2 = g(4k_1^2 + k_2^2)^{\frac{1}{2}}(1 + \epsilon_4). \tag{A 2}$$

Eliminating k_1 and k_2 from the above equations and neglecting squares and products of the ϵ_i we have

$$(2r-1)^4 = 4r^4(1-2\epsilon_1+2\epsilon_4) + (1-2\epsilon_2+2\epsilon_4),$$

where $r = \sigma_1/\sigma_2$, as before. Now write

$$r = r_0 + \Delta r,$$

where r_0 is the non-zero root of (2.5). Then to the same order of approximation we have

$$8(2r_0-1)^3\Delta r = 16r_0^3\Delta r - 8r_0^4(\epsilon_1-\epsilon_4) - 2(\epsilon_2-\epsilon_4),$$

and hence

$$\Delta r = \frac{4r_0^4\epsilon_1 + \epsilon_2 - (4r_0^4+1)\epsilon_4}{8r_0^3 - 4(2r_0-1)^3}.$$

Inserting the numerical value of r_0 from equation (2.5) we find

$$\Delta_r = -1{\cdot}957\epsilon_1 - 0{\cdot}054\epsilon_2 + 2{\cdot}011\epsilon_4. \tag{A 3}$$

(a) Correction for finite depth

The period equation for waves in water of uniform depth h is

$$\sigma^2 = gk\tanh kh.$$

Which can be written

$$\sigma^2 = gk(1-e^{-2kh})(1+e^{-2kh})^{-1}.$$

When e^{-kh} is small, this is replaceable by

$$\sigma^2 = gk(1-2e^{-2kh}). \tag{A 4}$$

(b) Correction for surface tension

For waves controlled by both gravity and surface tension we have

$$\sigma^2 = gk + (T/\rho)k^3,$$

where T is the surface-tension constant and ρ denotes the density. This may be written

$$\sigma^2 = gk\{1+(T/\rho g)k^2\}. \tag{A 5}$$

In the case of clean water at room temperature,

$$T/(\rho g) = 74/(1{\cdot}00\times 981) = 0{\cdot}075\,\text{cm}^2.$$

(c) *Correction for finite amplitude*

For waves of finite amplitude there is a small increase in the phase velocity, given by

$$\sigma^2 = gk(1+a^2k^2), \tag{A 6}$$

(see Lamb 1932, §250).

(d) *Correction for mutual interaction*

It was shown by Longuet-Higgins & Phillips (1962) that in the presence of any wave (wave-number $\mathbf{k}_1$), a second wave (wave-number $\mathbf{k}_2$) undergoes a slight change in phase velocity, and this change is given by

$$\Delta c_2 = K'/(2ga_2k_2),$$

where

$$K' = a_1^2a_2\sigma_1\sigma_2\Bigg[(\sigma_1-\sigma_2)|\mathbf{k}_1-\mathbf{k}_2|\{\cos^2\tfrac{1}{2}\theta\}\left\{1+\frac{4\sigma_1\sigma_2\sin^2\frac{1}{2}\alpha}{(\sigma_1-\sigma_2)^2-g|\mathbf{k}_1-\mathbf{k}_2|}\right\}$$
$$+(\sigma_1+\sigma_2)|\mathbf{k}_1+\mathbf{k}_2|\{\sin^2\tfrac{1}{2}\theta\}|\left\{1-\frac{4\sigma_1\sigma_2\sin^2\frac{1}{2}\beta}{(\sigma_1+\sigma_2)^2-g|\mathbf{k}_1+\mathbf{k}_2|}\right\}$$
$$+\sigma_1(k_1-k_2+k_2\cos^2\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta)+\sigma_2(k_1+k_2)\cos\theta\Bigg], \tag{A 7}$$

and α, β, θ are the angles shown in figure 7.

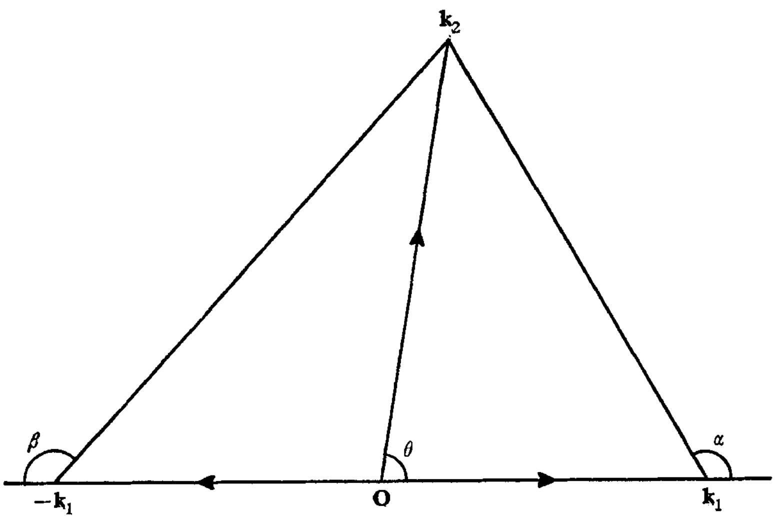

FIGURE 7. Definition diagram for the angles θ, α, β.

The proportional change in velocity is

$$\Delta c_2/c_2 = K'/(2ga_2\sigma_2),$$

and hence

$$\Delta(\sigma_1^2)/\sigma_2^2 = K'/(ga_2\sigma_2).$$

Thus,

$$\sigma_2^2 = gk_2\left(1+\frac{K'}{ga_2\sigma_2}\right),$$

or

$$\sigma_2^2 = gk_2(1+Ma_1^2k_1^2),$$

where

$$M = K'/(a_1^2a_2gk_1^2\sigma_2) = (\sigma_1k_1)^{-1}\times \text{square bracket in (A 7)}. \tag{A 8}$$

If the waves are mutually perpendicular ($\theta = \frac{1}{2}\pi$),

$$\sin^2 \tfrac{1}{2}\alpha = \sin^2 \tfrac{1}{2}\beta = \tfrac{1}{2}\left(1 + \frac{k_1}{(k_1^2 + k_2^2)^{\frac{1}{2}}}\right),$$

and we find after some reduction,

$$M = -\left[\frac{3}{4r^2} + \frac{1}{(r^4 - r^2 + 1) - (r^2 + 1)(r^4 + 1)^{\frac{1}{2}}}\right]. \tag{A 9}$$

At the resonant frequency, the effect of wave 1 on wave 2 is found by writing $r = r_0 = 1{\cdot}7357$ in (A 9) giving $M = -0{\cdot}073$. The effect of wave 2 on wave 1 is found by interchanging the suffices or inverting r, i.e. $r = 1 \div 1{\cdot}7357$, giving $M = -0{\cdot}66$.

The effect of wave 1 on wave 4 is found by replacing the suffix 2 by the suffix 4 in (A 9) and taking θ as the angle between $\mathbf{k}_1$ and $\mathbf{k}_3$; α and β being changed correspondingly. It is then found that $M = 2{\cdot}92$. Similarly, for the effect of wave 2 on wave 4 it is found that $M = -2{\cdot}15$, and for the effect of wave 4 on waves 1 and 2 we find $M = 0{\cdot}69$ and $-0{\cdot}06$, respectively.

These corrections are all summarized in table 1 (see p 420).

Appendix 2

The response function

We now calculate the amplitude of the tertiary wave at different places in the tank, and for different frequency ratios r, on the assumption that the waves are effectively deep-water gravity waves of small ampliutde, and ignoring the small corrections mentioned in Appendix 1.

If the surface elevation corresponding to the two primary wave trains is given by

$$a_1 \cos(\mathbf{k}_1 . \mathbf{x} - \sigma_1 t) + a_2 \cos(\mathbf{k}_2 . \mathbf{x} - \sigma_2 t),$$

then the equation for the tertiary wave potential can be shown (Longuet-Higgins 1962) to be as follows:

$$\left(\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right)_{z=0} = -K \sin[(2\mathbf{k}_1 - \mathbf{k}_2) . \mathbf{x} - (2\sigma_1 - \sigma_2) t], \tag{A 10}$$

where K is a function of the wave amplitudes a_1, a_2, the wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$ and the frequencies σ_1, σ_2. In the case we are considering, when the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$ is 90°, we have

$$K = \tfrac{1}{2}a_1^2 a_2 \sigma_1 \sigma_2 \left[(\sigma_1 - \sigma_2)\{k_1 + k_2 - |\mathbf{k}_1 - \mathbf{k}_2|\} + \tfrac{1}{2}\sigma_1 k_2 - \frac{4\sigma_1(\sigma_1 - \sigma_2)(2\sigma_1 - \sigma_2)|\mathbf{k}_1 - \mathbf{k}_2| \cos^2 \frac{1}{2}\alpha}{(\sigma_1 - \sigma_2)^2 - g|\mathbf{k}_1 - \mathbf{k}_2|}\right],$$

where α denotes the angle between $\mathbf{k}_1$ and $(\mathbf{k}_1 - \mathbf{k}_2)$. Write

$$|2\mathbf{k}_1 - \mathbf{k}_2| = k_4, \quad (2\sigma_1 - \sigma_2) = \sigma_4$$

and consider the solution to (A 10) together with Laplace's equation $\nabla^2 \phi = 0$ and the deep-water condition: $\phi \to 0$ as the vertical co-ordinate z tends to $-\infty$.

We also assume that the total flow across the plane $x = 0$ vanishes, where x is the horizontal co-ordinate in the direction of propagation. It can be verified that the solution is

$$\phi = \frac{K}{\sigma_4^2 - gk_4}[e^{k_4 z}\sin(k_4 x - \sigma_4 t) - e^{k_0 z}\sin(k_0 x - \sigma_4 t)], \tag{A 11}$$

where $k_0 = \sigma_4^2/g$. The first term in the square brackets represents a particular integral of (A 10), and the second term represents a free wave added in order to satisfy the imposed boundary condition at $x = 0$. Now when $z = 0$, (A 11) can be written,

$$\phi = \{-2K/(\sigma_4^2 - gk_4)\}\sin\tfrac{1}{2}(k_4 - k_0)x\sin\{\tfrac{1}{2}(k_4 + k_0)x - \sigma_4 t\},$$

or

$$\phi = -\frac{Kx}{g}\frac{\sin(x\delta k)}{(x\delta k)}\sin\{(k_0 + \delta k)x - \sigma_4 t\}, \tag{A 12}$$

where

$$\delta k = \frac{k_4 - k_0}{2} = \frac{1}{2}\left(k_4 - \frac{\sigma_4^2}{g}\right). \tag{A 13}$$

When $\delta k/k_4$ is small, (A 12) represents a wave of wave-number k_0 approximately, travelling with nearly the free-wave velocity. Its amplitude is given by

$$a_4 = \left|\frac{1}{g}\frac{\partial\phi}{\partial t}\right|_{z=0} = \frac{K\sigma x}{g^2}\left|\frac{\sin(x\delta k)}{(x\delta k)}\right|. \tag{A 14}$$

As the critical frequency ratio is approached, $\delta k \to 0$ and in the limit

$$a_4 = K\sigma x/g^2. \tag{A 15}$$

In other words, at resonance the amplitude of the tertiary wave grows proportionally to the distance x. Equation (A 15) may be written

$$a_4 = (a_1 k_1)^2 (a_2 k_2) G(\theta) x \tag{A 16}$$

(cf. Longuet-Higgins 1962, equation (7.2)), where G is a non-dimensional function of the angle θ between the two primary waves: in this case when $\theta = 90°$ it is found that $G = 0{\cdot}442\ldots$.

If now the ratio r of the frequencies is varied, equation (A 12) shows that the amplitude of the tertiary waves is given more generally by

$$a_4 = (a_1 k_1)^2 (a_2 k_2) G(\theta, r) x\left|\frac{\sin x\delta k}{x\delta k}\right|, \tag{A 17}$$

where δk is given by (A 13) and

$$G(\tfrac{1}{2}\pi, r) = \frac{2r-1}{r^3}\left[(r-1)\{(r^2+1) - (r^4+1)^{\frac{1}{2}}\} + \tfrac{1}{2}r + \frac{2r(r-1)(2r-1)\{(r^4+1)^{\frac{1}{2}} - r^2\}}{(r^4+1)^{\frac{1}{2}} - (r-1)^2}\right]. \tag{A 18}$$

The latter function is plotted in figure 8 over the interesting range of frequencies: $1{\cdot}4 \leqslant r \leqslant 2{\cdot}1$, and it will be seen that over the whole range $G(\frac{1}{2}\pi, r)$ differs little from $G(\frac{1}{2}\pi, r_0)$.

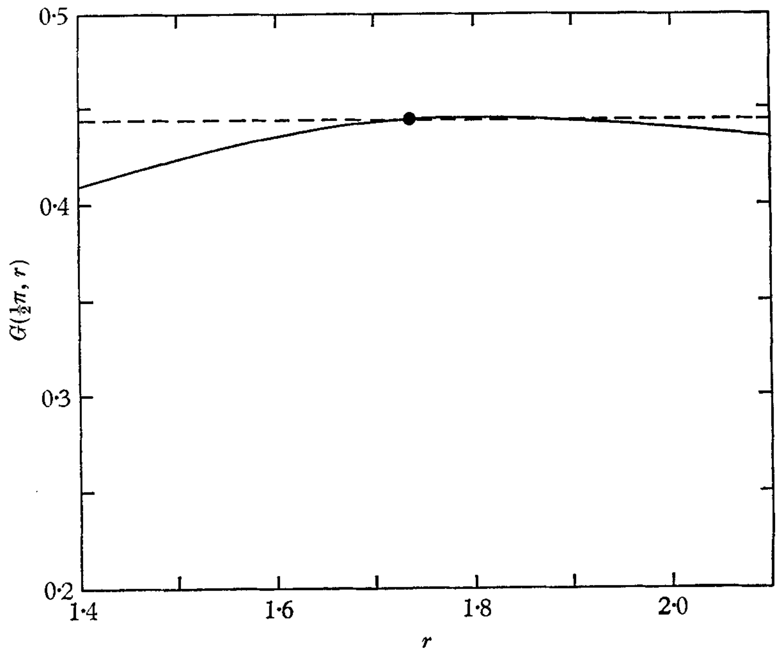

FIGURE 8. The function $G(\frac{1}{2}\pi, r)$ in $1 \cdot 4 \leqslant r \leqslant 2 \cdot 1$.
The broken line represents $G(\frac{1}{2}\pi, r_0)$.

We must now relate δk to the frequency ratio r. From the definition (A 13) we have

$$\frac{2\delta k}{k_4} = 1 - \frac{\sigma_4^2}{gk_4} = 1 - \frac{(2\sigma_1 - \sigma_2)^2}{g(4k_1^2 + k_2^2)^{\frac{1}{2}}}, \tag{A 19}$$

or from (2.1)

$$\frac{2\delta k}{k_4} = 1 - \frac{(2r-1)^2}{(4r^4+1)^{\frac{1}{2}}}. \tag{A 20}$$

If r_0 is the value corresponding to $\delta k = 0$, i.e. to resonance, and if $r = r_0 + \delta r$, then by differentiation of the right-hand side of (A 20) we have to first order,

$$\frac{2\delta k}{k_4} = -\left(\frac{4}{2r_0 - 1} - \frac{8r_0^3}{4r_0^4 + 1}\right)\delta r.$$

On inserting the numerical value of r_0 we find

$$2\delta k/k_4 = -0 \cdot 497 \delta r. \tag{A 21}$$

The exact expression (A 20) is compared with the approximation (A 21) in figure 9. So we have

$$\delta k = -0 \cdot 249 k_4 \delta r. \tag{A 22}$$

So far, the effect of viscosity on the waves has been entirely neglected. In that case it has been shown that at the critical ratio r_0, the amplitude of the waves tends to grow linearly with distance, i.e.

$$a_4 = Ax, \quad da_4/dx = A, \tag{A 23}$$

where

$$A = 0 \cdot 442 (a_1 k_1)^2 (a_2 k_2). \tag{A 24}$$

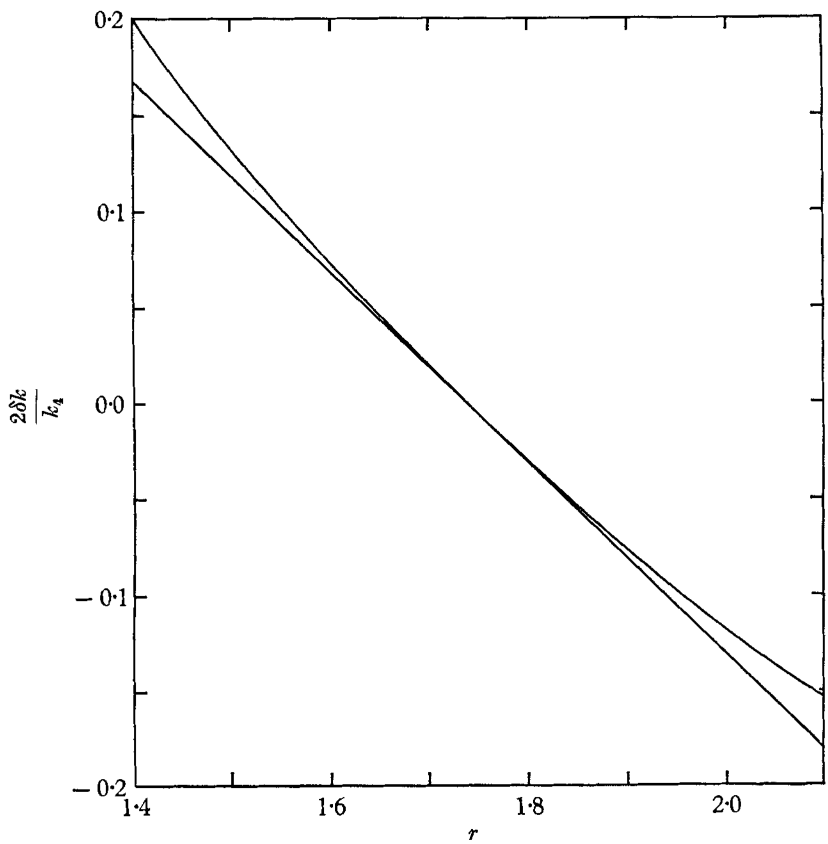

FIGURE 9. A graph of $2\delta k/k_4$, as given by (A 20), compared with the straight-line approximation (A 21).

We shall now consider the modification to this expression when viscosity is taken into account.

In a free-surface wave, the rate of dissipation of energy per unit time and horizontal distance is given by

$$2\rho\nu a^2\sigma^2 k$$

(see Lamb 1932). In gravity waves the energy $E = \frac{1}{2}\rho g a^2$ is propagated with the group velocity $\sigma/2k$, and hence we have the relation

$$\partial(\tfrac{1}{4}\rho g a_4^2\sigma/k)/\partial x = -2\rho\nu a_4^2\sigma^2 k.$$

Thus
$$\frac{1}{a_4}\frac{\partial a_4}{\partial x} = -4\nu\sigma^5/g^3 = -B, \tag{A 25}$$

say. Being proportional to the fifth power of the frequency, the damping increases very sharply towards the high frequencies.

We now attempt to combine (A 23) and (A 25). If the damping is small, it will have a first-order effect on the amplitude of the waves but only a second-order effect on the phase velocity. Hence the input of energy into the tertiary waves is unaffected by the viscosity†, though there is also a continuous loss of energy due to viscous dissipation. We assume then as a differential equation for a_4 the following:

$$da_4/dx = A - Ba_4, \tag{A26}$$

where A and B are the constants in equations (A24) and (A25). Equations (A23) and (A25) are the special cases when $B = 0$ and $A = 0$, respectively.

The solution of (A26) with boundary condition $a_4 = 0$ at $x = 0$, is

$$a_4 = A(1-e^{-Bx})/B. \tag{A27}$$

Clearly when $Bx \ll 1$ this reduces to (A23), i.e. the rate of growth is initially linear. However, when $Bx \gg 1$ we have the asymptotic value

$$a_4 \sim A/B,$$

which corresponds to the steady state, when the input of energy by the primary waves is exactly balanced by the dissipation of energy by viscosity.

REFERENCES

BENJAMIN, T. BROOKE & FEIR, J. 1966. The disintegration of wave-trains on deep water. Part 1. Theory. Part 2. Experiments. To appear in *J. Fluid Mech.*

BENNEY, D. J. 1962 Non-linear gravity-wave interactions. *J. Fluid Mech.* **14**, 577–84.

BRETHERTON, F. P. 1964 Resonant interactions between waves; the case of discrete oscillations. *J. Fluid Mech.* **20**, 457–479.

HASSELMANN, K. 1960 Grundgleichungen der Seegaugsvoraussage. *Schiffstechnik*, **7**, 191–5.

HASSELMANN, K. 1962 On the non-linear energy transfer in a gravity wave spectrum. Part 1: General theory. *J. Fluid Mech.* **12**, 481–500.

HASSELMANN, K. 1963*a* On the non-linear energy transfer in a gravity wave spectrum. Part II: Conservation theorems; wave particle analogy; irreversibility. *J. Fluid Mech.* **15**, 273–81.

HASSELMANN, K. 1963*b* On the non-linear energy transfer in a gravity wave spectrum. Part III: Evaluation of the energy flux and sea-swell interaction for a Neumann spectrum. *J. Fluid Mech.* **15**, 385–98.

LAMB, H. 1932 *A Treatise on Hydrodynamics*, 6th ed. Cambridge University Press.

LONGUET-HIGGINS, M. S. 1960 Mass-transport in the boundary layer at a free oscillating surface. *J. Fluid Mech.* **8**, 293–306.

LONGUET-HIGGINS, M. S. 1962 Resonant interactions between two trains of gravity waves. *J. Fluid Mech.* **12**, 321–32.

LONGUET-HIGGINS, M. S. & PHILLIPS, O. M. 1962 Phase velocity effects in tertiary wave interactions. *J. Fluid Mech.* **12**, 333–6.

MCGOLDRICK, L. F., PHILLIPS, O. M., HUANG, N. E. & HODGSON, T. H. 1966 Measurements of third-order resonant wave interactions. *J. Fluid Mech.* **25**, 437.

PHILLIPS, O. M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. Part I. The elementary interactions. *J. Fluid Mech.* **9**, 193–217.

PIERSON, W. J. 1961 Comments on a paper by O. M. Phillips. *Proc. Conf. on Ocean Wave Spectra*, Easton Md. 1961; Prentice-Hall 1963. See pp. 180–6 and 187–8.

† The effect of viscosity on the two primary wave trains is even smaller, owing to their greater wavelength.